Topological Structures on Intuitionistic Fuzzy Multisets

Shinoj T K, Sunil Jacob John

Abstract – In this paper, we introduced the concept of Intuitionistic Fuzzy Multiset Topology. Mapping functions are defined to connect Intuitionistic Fuzzy Multisets defined on different sets. Subspaces and Continuous functions are discussed to study the topological structures of Intuitionistic Fuzzy Multisets and their various properties are discussed.

Keywords-- Intuitionistic Fuzzy Multiset, Intuitionistic Fuzzy Multiset Topology, Continuous functions, Subspaces.

I. INTRODUCTION

THE theory of sets, one of the most powerful tools in modern mathematics is usually considered to have begun with Georg Cantor (1845-1918). Considering the uncertainty factor, Lotfi A. Zadeh [1] introduced Fuzzy sets in 1965, in which a membership function assigns to each element of the universe of discourse, a number from the unit interval [0,1] to indicate the degree of belongingness to the set under consideration.

If repeated occurrences of any object are allowed in a set, then the mathematical structure is called as multiset [11,12]. As a generalization of multiset, Yager [2] introduced fuzzy multisets and suggested possible applications to relational databases. An element of a Fuzzy Multiset can occur more than once with possibly the same or different membership values.

In 1983, Atanassov [3,10] introduced the concept of Intuitionistic Fuzzy sets. An Intuitionistic Fuzzy set is characterized by two functions expressing the degree of membership and the degree of nonmembership of elements of the universe to the Intuitionistic Fuzzy set. Among the various notions of higher-order Fuzzy sets, Intuitionistic Fuzzy sets proposed by Atanassov provide a flexible framework to explain uncertainty and vagueness.

The concept of Intuitionistic Fuzzy Multiset is introduced in [4] by combining the all the above concepts. Intuitionistic Fuzzy Multiset has applications in medical diagnosis and robotics [13,14].

In [5] Shinoj T K, Anagha Baby and Sunil Jacob John introduced algebraic structures on Intuitionistic Fuzzy Multiset.

General topology was the first field of pure mathematics where the concepts and ideas of fuzzy sets took strong roots. In 1968, Chang [9] introduced Fuzzy topological spaces. And as a continuation of this, in 1997, Dogan Coker [6] introduced the concept of Intuitionistic fuzzy topological spaces. In our work, we generalized this concept into Intuitionistic Fuzzy Multiset. First we discuss the basic operations and in the subsequent sections we introduce the concept of functions on Intuitionistic Fuzzy Multiset and followed by this the topological structures and its various properties are discussed.

• *Shinoj T K. E-mail: shinojthrissur@gmail.com. Department of Mathematics, National Institute of Technology,Calicut-673 601, Kerala, India.*
• *Sunil Jacob John: sunil@nitc.ac.in. Department of Mathematics, National Institute of Technology,Calicut-673 601, Kerala, India.*

II. PRELIMINARIES

2.1 Definition **[1]** Let *X* be a nonempty set. A *Fuzzy set A* drawn from *X* is defined as $A = \{< x : \mu_A(x) > : x \in X\}$.Where : $X \to [0,1]$ is the membership function of the Fuzzy Set *A*.

2.2. *Definition* **[2]** Let X be a nonempty set. A *Fuzzy Multiset* (FMS) A drawn from X is characterized by a function, 'count membership' of A denoted by CM_A such that $CM_A : X \to Q$ where Q is the set of all crisp multisets drawn from the unit interval [0,1]. Then for any $x \in X$, the value $CM_A(x)$ is a crisp multiset drawn from [0,1]. For each $x \in X$, the membership sequence is defined as the decreasingly ordered sequence of elements in $CM_A(x)$. It is denoted by $(\mu^1_A(x), \mu^2_A(x),...,\mu^p_A(x))$ where $\mu^1_A(x) \geq \mu^2_A(x) \geq ,..., \geq \mu^p_A(x)$.

A complete account of the applications of Fuzzy Multisets in various fields can be seen in [9].

2.3 Definition **[3]** Let X be a nonempty set. An *Intuitionistic Fuzzy Set* (IFS) A is an object having the form $A = \{< x: \mu_A(x), \nu_A(x) >: x \in X\}$, where the functions $\mu_A: X \to [0,1]$ and $\nu_A: X \to [0,1]$ define respectively the degree of membership and the degree of non membership of the element $x \in X$ to the set A with $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$.

2.4 Remark Every Fuzzy set A on a nonempty set X is obviously an IFS having the form

$$A = \{< x : \mu_A(x), 1 - \mu_A(x) > : x \in X\}$$

Using the definition of FMS and IFS, a new generalized concept can be defined as follows:

***2.5 Definition* [4]** Let X be a nonempty set. An *Intuitionistic Fuzzy Multiset* A denoted by IFMS drawn from X is characterized by two functions : 'count membership' of A (CM_A) and 'count non membership' of A (CN_A) given respectively by $CM_A : X \rightarrow Q$ and $CN_A : X \rightarrow Q$ where Q is the set of all crisp multisets drawn from the unit interval [0, 1] such that for each $x \in X$, the membership sequence is defined as a decreasingly ordered sequence of elements in CMA(x) which is denoted by $(\mu^1_A(x), \mu^2_A(x),...,\mu^P_A(x))$ where $(\mu^1_A(x) \geq \mu^2_A(x) \geq,... \geq,\mu^P_A(x)$ and the corresponding non membership sequence will be denoted by $(\nu^1_A(x), \nu^2_A(x),...,\nu^P_A(x))$ such that $0 \leq \mu^i_A(x) + \nu^i_A(x) \leq 1$ for every $x \in X$ and i = 1,2,...,p.

An IFMS A is denoted by
$A = \{< x : (\mu^1_A(x), \mu^2_A(x),...,\mu^P_A(x)), (\nu^1_A(x), \nu^2_A(x), ... ,\nu^P_A(x)) > : x \in X\}$

2.6 Remark We arrange the membership sequence in decreasing order but the corresponding non membership sequence may not be in decreasing or increasing order.

***2.7 Definition* [4]** For any two IFMSs A and B drawn from a set X, the following operations and relations will hold. Let $A = \{< x : (\mu^1_A(x), \mu^2_A(x),...,\mu^P_A(x)), (\nu^1_A(x), \nu^2_A(x),...,\nu^P_A(x)) > : x \in X\}$ and

$B = \{< x : (\mu^1_B(x), \mu^2_B(x),...,\mu^P_B(x)), (\nu^1_B(x), \nu^2_B(x),...,\nu^P_B(x)) > : x \in X\}$ then

1. Inclusion

   $A \subset B \Leftrightarrow \mu^j_A(x) \leq \mu^j_A(x)$ and $\nu^j_A(x) \geq \nu^j_B(x)$;

   $j = 1,2,..., L(x), x \in X$

   $A = B \Leftrightarrow A \subset B$ and $B \subset A$

2. Complement

   $\nabla A = \{< x : (\nu^1_A(x),...,\nu^P_A(x)), (\mu^1_A(x),....... ...,\mu^P_A(x)) > : x \in X\}$

3. Union $(A \cup B)$
   In $A \cup B$ the membership and non membership values are obtained as follows.

   $\mu^j_{A\cup B}(x) = \mu^j_A(x) \vee \mu^j_B(x)$

   $\nu^j_{A\cup B}(x) = \nu^j_A(x) \wedge \nu^j_B(x)$

   $j = 1, 2,...,L(x), x \in X.$

4. Intersection $(A \cap B)$
   In $A \cap B$ the membership and non membership values are obtained as follows.

   $\mu^j_{A\cap B}(x) = \mu^j_A(x) \wedge \mu^j_B(x)$

   $\nu^j_{A\cap B}(x) = \nu^j_A(x) \vee \nu^j_B(x)$

   $j = 1, 2,...,L(x), x \in X.$

5. Addition $(A \oplus B)$
   In $A \oplus B$ the membership and non membership values are obtained as follows.
   $\mu^j_{A\oplus B}(x) = \mu^j_A(x) + \mu^j_B(x) - \mu^j_A(x). \mu^j_B(x)$
   $\nu^j_{A\oplus B}(x) = \nu^j_A(x). \nu^j_B(x)$
   $j = 1, 2,...,L(x), x \in X.$

6. Multiplication $(A \otimes B)$
   In $A \otimes B$ the membership and non membership values are obtained as follows.
   $\mu^j_{A\otimes B}(x) = \mu^j_A(x). \mu^j_B(x)$
   $\nu^j_{A\otimes B}(x) = \nu^j_A(x) + \nu^j_B(x) - \nu^j_A(x). \nu^j_B(x)$
   $j = 1, 2,...,L(x), x \in X.$

here $\vee, \wedge, ., +, -$ denotes maximum, minimum, multiplication, addition, subtraction of real numbers respectively.

***2.8 Definition* [5]** Let X and Y be two nonempty sets and $f : X \rightarrow Y$ be a mapping. Then

a) The image of the FMS $A \in FM(X)$ under the mapping f is denoted by $f(A)$ or $f[A]$, where

$$CM_{f[A]}(y) = \begin{cases} \vee_{f(x)=y} CM_A(x) \; ; & f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

b) The inverse image of the FMS $B \in FM(Y)$ under the mapping f is denoted by $f^{-1}(B)$ or $f^{-1}[B]$, where $CM_{f^{-1}[B]}(x) = CM_B f[x]$.

ISSN 2229-5518

***2.9 Definition* [6]** Let X and Y be two nonempty sets and $f: X \to Y$ be a mapping. Then

a) If B = {< y, μ_B(y), ν_B(y) >: y ϵ Y} is an IFS in Y, then the preimage of B under f, denoted by f^{-1}(B), is the IFS in X defined by

$$f^{-1}(B) = \{< x, f^{-1}(\mu_B)(x), f^{-1}(\nu_B)(x)) >: x \epsilon X\}$$

b) If A = {< x, λ_A(x), γ_A(x) >: x ϵ X} is an IFS in X, then the image of A under f, denoted by f(A), is the IFS in Y defined by,

$f(A) = \{< y, f(\lambda_A)(y), f_{-1}(\gamma_A)(y)) >: y \epsilon Y\}$ where

$$f(\lambda_A)(y) = \begin{cases} \sup_{x\epsilon f^{-1}(y)} \lambda_A(x) \ ; & f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$f_{-1}(\gamma_A)(y) = \begin{cases} \inf_{x\epsilon f^{-1}(y)} \gamma_A(x) \ ; & f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

In the next section we amalgamate the above two definitions of functions on FMS and IFS to define the functions on IFMS.

III. FUNCTIONS ON INTUITIONISTIC FUZZY MULTISETS

3.1 Definition Let X and Y be two nonempty sets and $f: X \to Y$ be a mapping. Then

a) The image of an IFMS A in X under the mapping f is denoted by f(A) is defined as

$$CM_{f[A]}(y) = \begin{cases} \vee_{f(x)=y} CM_A(x) \ ; & f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$CN_{f[A]}(y) = \begin{cases} \wedge_{f(x)=y} CN_A(x) \ ; & f^{-1}(y) \neq \emptyset \\ 1 & \text{otherwise} \end{cases}$$

b) The inverse image of the IFMS B in Y under the mapping f is denoted by $f^{-1}(B)$ where

$$CM_{f^{-1}[B]}(x) = CM_B f[x], CN_{f^{-1}[B]}(x) = CN_B f[x]$$

In the next proposition we discuss some of the properties of functions which we have proved for FMS in [5].

3.2 Proposition Let X, Y and Z be three nonempty sets and $f: X \to Y$ and $g: Y \to Z$ be two mappings. If A, A_i ∈ IFMS(X), B, B_i ∈ IFMS(Y), C ∈ IFMS(Z) ; i ∈ I then

a) $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$

b) $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$

c) $f[\cup_{i\in I} A_i] = \cup_{i\in I} f[A_i]$

d) $f^{-1}[\cup_{i\in I} B_i] = \cup_{i\in I} f^{-1}[B_i]$

e) $f^{-1}[\cap_{i\in I} B_i] = \cap_{i\in I} f^{-1}[B_i]$

f) $g[f(A_i)] = [gf](A_i)$ and $f^{-1}[g^{-1}(B_j)] = [gf]^{-1}(B_j)$

Proof:

a) Let $A_1 \subseteq A_2$.

Then $CM_{A_1}(x) \leq CM_{A_2}(x) \ \forall \ x \in X$

$\Rightarrow \vee_{f(x)=y} CM_{A_1}(x) \leq \vee_{f(x)=y} CM_{A_2}(x)$

$\Rightarrow CM_{f(A_1)}(y) \leq CM_{f(A_2)}(y) \ \forall y = f(x)$ …………………(1)

Also $CN_{A_1}(x) \geq CN_{A_2}(x) \ \forall \ x \in X$

$\Rightarrow \wedge_{f(x)=y} CN_{A_1}(x) \geq \wedge_{f(x)=y} CN_{A_2}(x)$

$\Rightarrow CN_{f(A_1)}(y) \geq CN_{f(A_2)}(y) \ \forall y = f(x)$ …………………(2)

(1) and (2) $\Rightarrow f(A_1) \subseteq f(A_2)$

b) Let $B_1 \subseteq B_2$

$\Rightarrow CM_{B_1}(f(x)) \leq CM_{B_2}(f(x)$ and $CN_{B_1}(f(x)) \geq CN_{B_2}(f(x))$

$\Rightarrow CM_{f^{-1}(B_1)}(x) \leq CM_{f^{-1}(B_2)}(x)$ and $CN_{f^{-1}(B_1)}(x) \geq CN_{f^{-1}(B_2)}(x)$ by 3.1(b)

$\Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$.

c) Let $A = \bigcup A_i$

So $CM_{f(\cup_{i\in I} A_i)}(y) = CM_{f(A)}(y)$

$= \vee_{f(x)=y} CM_A(x)$

$= \vee_{f(x)=y}\{\vee_{i\in I} CM_{A_i}(x)\}$

$= \vee_{i\in I}\{\vee_{f(x)=y} CM_{A_i}(x)\}$

$= \vee_{i\in I} CM_{f(A_i)}(y)$

$= CM_{\cup_{i\in I}[f(A_i)]}(y)$ ………………(1)

And $CN_{f(\cup_{i\in I} A_i)}(y) = CN_{f(A)}(y)$

$= \wedge_{f(x)=y} CN_A(x)$

$= \wedge_{f(x)=y}\{\wedge_{i\in I} CN_{A_i}(x)\}$

$= \wedge_{i\in I}\{\wedge_{f(x)=y} CN_{A_i}(x)\}$

$= \wedge_{i\in I} CN_{f(A_i)}(x)\}$

$= CN_{\cup_{i\in I}[f(A_i)]}(y)$ ………………(2)

From (1) and (2) $f[\cup_{i\in I} A_i] = \cup_{i\in I} f[A_i]$

d) Let $\cup_{i\in I} B_i = B$

$CM_{f^{-1}(\cup_{i\in I} B_i)}(x) = CM_{f^{-1}(B)}(x)$

$= CM_B f(x)$

$= \vee_{i\in I} CM_{B_i} f(x)$

$= \vee_{i\in I} CM_{f^{-1}[B_i]}(x)$

ISSN 2229-5518

$= CM_{\cup_{i\in I} f^{-1}(B_i)}(x)$

…………………(1)

And $CN_{f^{-1}(\cup_{i\in I} B_i)}(x) = CN_{f^{-1}(B)}(x)$

$= CN_B f(x)$

$= \wedge_{i\in I} CN_{B_i} f(x)$

$= \wedge_{i\in I} CN_{f^{-1}(B_i)} x$

$= CN_{\cup_{i\in I} f^{-1}(B_i)}(x)$

…………………(2)

So $f^{-1}[\cup_{i\in I} B_i] = \cup_{i\in I} f^{-1}[B_i]$.

e) Let $\cap_{i\in I} B_i = B$

$CM_{f^{-1}(\cap_{i\in I} B_i)}(x) = CM_{f^{-1}(B)}(x)$

$= CM_B f(x)$

$= \wedge_{i\in I} CM_{B_i} f(x)$

$= \wedge_{i\in I} CM_{f^{-1}(B_i)}(x)$

$= CM_{\cap_{i\in I} f^{-1}(B_i)}(x)$

…………………(1)

And $CN_{f^{-1}(\cap_{i\in I} B_i)}(x) = CN_{f^{-1}(B)}(x)$

$= CN_B f(x)$

$= \vee_{i\in I} CN_{B_i} f(x)$

$= \vee_{i\in I} CN_{f^{-1} B_i} x$

$= CN_{\cap_{i\in I} f^{-1}(B_i)}(x)$

…………………(2)

So $f^{-1}[\cap_{i\in I} B_i] = \cap_{i\in I} f^{-1}[B_i]$

f) Let $A \in IFMS(X)$ and $z \in Z$

Then $CM_{g[f(A)]}(z) = \vee_{g(y)=z} CM_{f(A)}(y)$; $y \in Y$

$= \vee_{g(y)=z}\{\vee_{f(x)=y} CM_A(x)\}$; $y \in Y$ and $x \in X$

$= \vee\{\vee\{CM_A(x) ; x \in X \text{ and } f(x) = y\} ; y \in Y \text{ and } g(y) = z\}$

$= \vee\{CM_A(x) ; x \in X \text{ and } [gf](x) = z\}$

$= \vee_{[gf](x)=z} CM_A(x)$; $x \in X$

$= CM_{[gf](A)}(z)$.

…………………(1)

And $CN_{g[f(A)]}(z) = \wedge_{g(y)=z} CN_{f(A)}(y)$; $y \in Y$

$= \wedge_{g(y)=z}\{\wedge_{f(x)=y} CN_A(x)\}$; $y \in Y$ and $x \in X$

$= \wedge\{\wedge\{CN_A(x) ; x \in X \text{ and } f(x) = y\} ; y \in Y \text{ and } g(y) = z\}$

$= \wedge\{CN_A(x) ; x \in X \text{ and } [gf](x) = z\}$

$= \wedge_{[gf](x)=z} CN(x)$; $x \in X$

$= CN_{[gf](A)}(z)$.

………………(2)

(1) and (2) $\Rightarrow$ $g[f(A)] = [gf](A)$.

Using the first part and 3.1(b) we can prove the second part.

IV. INTUITIONISTIC FUZZY MULTISET TOPOLOGICAL SPACES

In this section we introduced the concept of Intuitionistic Fuzzy multiset Topology (IFMT). Here we extend the concept of Intuitionistic fuzzy topological spaces introduced by Dogan Coker in [6] to the case of Intuitionistic fuzzy multisets.

For this first we introduced $\neg 0$ and $\neg 1$ in a nonempty set X as follows.

4.1Definition

Let $\neg 0 = \{< x: (0,0,\ldots,0), (1,1,\ldots,1): x \in X\}$

$\neg 1 = \{< x: (1,1,\ldots 1), (0,0,\ldots,0) : x \in X\}$

4.2 Definition An intuitionistic Fuzzy multiset topology (IFMT) on X is a family τ of intuitionistic fuzzy multisets (IFMSs) such that

1. $\neg 0, \neg 1 \in \tau$
2. $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$
3. $\cup G_i \in \tau$ for any arbitrary family $\{G_i : i \in I\}$ in τ

Then the pair (X, τ) is called **Intuitionistic Fuzzy multiset topological space** (IFMT for short) and any IFMS in τ is known as an open intuitionistic fuzzy multiset (OIFMS in short) in X.

4.3 Remark The complement of an OIFMS is called closed intuitionistic Fuzzy multiset (CIFMS in short)

4.4 Example Let X = {1, 2} and define the IFMSs in X as follows.

For $n \in N^+$, $p \in N$

$G_n = \{<1: (n/n+1, n+1/n+2, \ldots, n+p/n+p+1), (1/n+2, 1/n+3, \ldots, 1/n+p+2)>,$

$<2: (n+1/n+2, n+2/n+3, \ldots, n+p+1/n+p+2), (1/n+3, 1/n+4, \ldots, 1/n+p+3)> \}$

Let $\tau = \{\neg 0, \neg 1\} \cup \{G_n\}$

Then (X, τ) forms an IFMT. Note that here membership values forms a monotonically increasing sequence and nonmembership values forms a monotonically decreasing sequence.

Construction of IFMTs

Here we construct Intuitionistic fuzzy multiset topology from a given IFMT. Dogan Coker in [6] has constructed these topologies for IFS.

Consider a nonempty set X. Let A = {< x : ($\mu^1{}_A(x)$, $\mu^2{}_A(x),\ldots,\mu^p{}_A(x)$), ($v^1{}_A(x)$, $v^2{}_A(x),\ldots,v^p{}_A(x)$) > : x ϵ X} be an IFMS. Define

[]A = {< x : ($\mu^1{}_A(x)$, $\mu^2{}_A(x),\ldots,\mu^p{}_A(x)$),(1-$\mu^1{}_A(x)$,1-$\mu^2{}_A(x)$,…,1-$\mu^p{}_A(x)$) > : x ϵ X}

4.5 Proposition Let (X,ɼ) be an IFMT on X. Then $ɼ_{0,1}$ = {[]A: A ϵ ɼ} is an IFMS.

Proof

It is obvious that ¬0, ¬1ϵ $ɼ_{0,1}$.

Let A_1, A_1 ϵ $ɼ_{0,1}$.

Then A_1 = {[]A_1'; A_1' ϵ ɼ} and A_2 = {[]A_2'; A_2' ϵ ɼ}

⟹ A_1 = {< x : ($\mu^1{}_{A1'}(x)$, $\mu^2{}_{A1'}(x)$,…, $\mu^p{}_{A1'}(x)$),(1- $\mu^1{}_{A1'}(x)$,1-$\mu^2{}_{A1'}(x)$,…,1- $\mu^p{}_{A1'}(x)$) > : x ϵ X} and

A_2 = {< x : ($\mu^1{}_{A2'}(x)$, $\mu^2{}_{A2'}(x)$,…, $\mu^p{}_{A2'}(x)$),(1- $\mu^1{}_{A2'}(x)$,1-$\mu^2{}_{A2'}(x)$,…,1- $\mu^p{}_{A2'}(x)$) > : x ϵ X}

Then $A_1 \cap A_2$ = {< x : ($\mu^1{}_{A1'}(x) \wedge \mu^1{}_{A2'}(x)$,…, $\mu^p{}_{A1'}(x) \wedge \mu^p{}_{A2'}(x)$),((1- $\mu^1{}_{A1'}(x)$) ∨ (1- $\mu^1{}_{A2'}(x)$,…,1- $\mu^p{}_{A1'}(x)$ ∨ (1- $\mu^p{}_{A2'}(x)$) > : x ϵ X}

[]($A_1' \cap A_2'$) = []{< x : ($\mu^1{}_{A1'}(x) \wedge \mu^1{}_{A2'}(x)$,…, $\mu^p{}_{A1'}(x) \wedge \mu^p{}_{A2'}(x)$),($v^1{}_{A1'}(x) \vee v^1{}_{A2'}(x)$,…, $v^p{}_{A1'}(x) \vee v^p{}_{A2'}(x)$) > : x ϵ X}

Since 1- {$\mu^i{}_{A1'}(x) \wedge \mu^i{}_{A2'}(x)$} = {1- $\mu^i{}_{A1'}(x)$} ∨ {1- $\mu^i{}_{A2'}(x)$}

$A_1 \cap A_2$ = []($A_1' \cap A_2'$) ϵ $ɼ_{0,1}$

Now If A_i ϵ $ɼ_{0,1}$; i ϵ I

Then A_i = {[]A_i'; A_i' ϵ ɼ}

UA_i = U([]A_i') = [](UA_i') ϵ $ɼ_{0,1}$.

Hence the proof.

Closure and Interior

4.6 Definition Let (X, ɼ) be an IFMT and A be an IFMS in X. Then **closure** of A denoted by cl(A) is defined as cl(A) = ∩{M: M is closed in X and A ⊆ M}.

4.7 Definition Let (X, ɼ) be an IFMT and B be an IFMS in X. Then **interior** of B is denoted by

int(B) is defined as int(B) = U{N: N is open in X and N ⊆ B}.

4.8 Definition Let (X, $ɼ_1$) and (X, $ɼ_2$) be two IFMTs on X. Then $ɼ_1$ is **coarser (weaker)** than $ɼ_2$ if Aϵ$ɼ_2$ for each Aϵ$ɼ_1$. It is denoted as $ɼ_1 \subseteq ɼ_2$.

4.9 Proposition Let {$ɼ_i$: I ϵ I} be a family of IFMTs on X. Then ∩$ɼ_i$ is an IFMT on X and it is the coarsest IFMT on X containing all the $ɼ_i$'s.

Proof follows from the definitions 4.2 and 4.8.

4.10 Proposition Let (X, ɼ) be an IFMT and A be an IFMS in X. Then cl(A) is a CIFMS.

Proof:

cl(A) = ∩{M: M is closed in X and A ⊆ M}

⟹ ∇cl(A) is the union of all open sets and hence it is open.

⟹ cl(A) is a CIFMS.

4.11 Proposition Let (X, ɼ) be an IFMT and A be an IFMS in X. Then int(A) is an OIFMS.

Proof:

Since int(A) is the union of all OIFMS which is contained in A, the proof follows from definition 4.2.3.

From the above two propositions and by the definitions it is clear that cl(A) is the smallest CIFMS which contains A and int(A) is the largest open set which is contained in A.

4.12 Proposition Let (X, ɼ) be an IFMT and A be an IFMS. Then cl(∇A) = ∇(int(A))

Proof:

Let A = {< x : ($\mu^1{}_A(x),\mu^2{}_A(x),\ldots,\mu^p{}_A(x)$),(($v^1{}_A(x),v^2{}_A(x),\ldots,v^p{}_A(x)$) >) > : x ϵ X}

Let A_i = {< x : ($\mu^1{}_{Ai}(x)$, $\mu^2{}_{Ai}(x),\ldots,\mu^p{}_{Ai}(x)$),(($v^1{}_{Ai}(x),v^2{}_{Ai}(x),\ldots,v^p{}_{Ai}(x)$) >) > : x ϵ X, i ϵ I} be the family of OIFMSs which is contained in A.

Then $\mu^j_{A_i} \leq \mu^j_A$ and $\nu^j_{A_i} \geq \nu^j_A$; $i \in I$, $j = 1,2,\ldots,p$ …………(1)

And
$$\text{int}(A) = \{< x: (\vee\mu^1{}_{A_i}(x), \ldots, \vee\mu^p{}_{A_i}(x)), (\wedge\nu^1{}_{A_i}(x), \ldots, \wedge\nu^p{}_{A_i}(x)) >: x \in X\}$$

$$\Rightarrow \nabla(\text{int}(A)) = \{< x: \left(\wedge\nu^1{}_{A_i}(x), \ldots, \wedge\nu^p{}_{A_i}(x)\right), (\vee\mu^1{}_{A_i}(x), \ldots, \vee\mu^p{}_{A_i}(x)) >: x \in X\}$$ ……(2)

Now $\nabla A = \{< x : ((\nu^1{}_A(x),\nu^2{}_A(x),\ldots,\nu^p{}_A(x)), (\mu^1{}_A(x),\mu^2{}_A(x),\ldots, \mu^p{}_A(x)) > : x \in X\}$

Also from (1) it is clear that $\{\nabla A_i;\ i \in I\}$ is the family of CIFMTs containing ∇A.

$$\Rightarrow cl(\nabla A) = \{< x: \left(\wedge\nu^1{}_{A_i}(x), \ldots, \wedge\nu^p{}_{A_i}(x)\right), (\vee\mu^1{}_{A_i}(x), \ldots, \vee\mu^p{}_{A_i}(x)) >: x \in X\}$$ …….(3)

From (2) and (3) $cl(\nabla A) = \nabla(\text{int}(A))$.

4.13 Proposition Let (X, Γ) be an IFMT and A be an IFMS. Then $\text{int}(\nabla A) = \nabla(cl(A))$

Proof is similar to proposition 4.12.

The following properties can be easily derived from the definitions.

4.14 Proposition Let (X, Γ) be an IFMT and A and B are OIFMSs. Then the following properties hold.

I. $\text{int}(A) \subseteq A$
II. $A \subseteq cl(A)$
III. $A \subseteq B \Rightarrow \text{int}(A) \subseteq \text{int}(B)$
IV. $A \subseteq B \Rightarrow cl(A) \subseteq cl(B)$
V. $\text{int}(\neg 1) = \neg 1$
VI. $cl(\neg 0) = \neg 0$

4.15 Proposition Let (X, Γ) be an IFMT and A be an IFMS in X. Then A is a CIFMS if and only if cl(A) = A.

Proof:

Assume that A is a CIFMS.

From 4.14.2, $A \subseteq cl(A)$ ………..(1)

Since A is a CIFMS, $cl(A) \subseteq A$ ………..(2)

From (1) and (2) cl(A) = A.

Now assume the converse. Hence by proposition 4.10, A is a CIFMS.

Hence the proof.

In the same way we can prove the next proposition.

4.16 Proposition Let (X, Γ) be an IFMT and A be an IFMS in X. Then A is an OIFMS if and only if int(A) = A.

Continuous Functions

4.17 Definition Let (X, Γ) and (Y, ϕ) be two IFMTs. A function f : X →Y is said to be **Continuous** if and only if inverse image of each OIFMS in ϕ is an OIFMS in Γ.

Next we discuss some of the equivalence relations of Continuous functions.

4.18 Theorem Let (X, Γ) and (Y, ϕ) be two IFMTs. Then the function f : X →Y is Continuous if and only if inverse image of each CIFMS in ϕ is a CIFMS in Γ.

Proof:

Assume that f is continuous and C be a CIFMS in ϕ.

To prove $f^{-1}(C)$ is closed, it is enough to show that $\neg f^{-1}(C)$ is an OIFMS in Γ.

Now the inverse image of the IFMS C in Y under the mapping f is denoted by $f^{-1}(C)$ where

$$CM_{f^{-1}[C]}(x) = CM_C f[x], CN_{f^{-1}[C]}(x) = CN_C f[x]$$

$$\Rightarrow f^{-1}(C) = \{x: < CM_C f[x], CN_C f[x] >\}$$

$$\Rightarrow \neg f^{-1}(C) = \{x: < CN_C f[x], CM_C f[x] >\}$$ …………..(1)

Now $f^{-1}(\neg C) = f^{-1}\{y: < CN_C(y), CM_C(y) >\}$

$$\Rightarrow f^{-1}(\neg C) = \{x: < CN_C f[x], CM_C f[x] >\}$$ …………..(2)

Where y = f(x)

From (1) and (2)

$\neg f^{-1}(C) = f^{-1}(\neg C)$ …………..(3)

Since f is continuous and $\neg C$ is open, definition 4.15 $f^{-1}(\neg C)$ is open and hence $f^{-1}(C)$ is closed.

Now assume that inverse image of each CIFMS in ф is a CIFMS in r.

To prove f is continuous, it is enough to prove $f^{-1}(O)$ is open for every OIFMS O in ф.

O is an OIFMS ⇒ ¬O is a CIFMS.

⇒ $f^{-1}(\neg O)$ is a CIFMS. (By assumption)

Since (3) is true for any OIFMS, $\neg f^{-1}(O)$ is a CIFMS and hence $f^{-1}(O)$ is an OIFMS.

Thus f is continuous.

Hence the proof.

4.19 Theorem Let (X, r) and (Y, ф) be two IFMTs. Then the function f : X →Y is Continuous if and only if for each IFMT A in X, f[cl(A)] ⊆ cl[f(A)]

Proof:

Assume f is continuous.

For any IFMT A in X, f(A) ⊆ cl(f(A)) by proposition 4.14.2

⇒ $A = f^{-1}(f(A)) \subseteq f^{-1}(cl[f(A)])$ by proposition 3.2.b

Since f is continuous and cl[f(A)] is closed, $f^{-1}(cl[f(A)])$ is closed.

⇒ $cl(A) \subseteq f^{-1}(cl[f(A)])$, since cl(A) is the smallest CIFMS contains A.

⇒ f(cl(A)) ⊆ cl[f(A)].

Conversely assume the given condition.

To prove f is continuous, let C be a CIFMS in ф.

Then by assumption, $f(cl(f^{-1}(C))) \subseteq cl[f(f^{-1}(C))] = cl(C) = C$, By proposition 4.15.

Thus $cl(f^{-1}(C)) \subseteq f^{-1}(C)$

But by proposition 4.14.2, $f^{-1}(C) \subseteq cl(f^{-1}(C))$

⇒ $f^{-1}(C) = cl(f^{-1}(C))$

⇒ $f^{-1}(C)$ is closed, By proposition 4.10.

⇒ f is continuous.

Hence the proof.

4.20 Theorem Let (X, r) and (Y, ф) be two IFMTs. Then the function f: X →Y is Continuous if and only if for each IFMT B in Y, $cl[f^{-1}(B)] \subseteq f^{-1}[cl(B)]$

Proof:

Replace A by $f^{-1}(B)$ in theorem 4.19.

4.21 Theorem Let (X, r) and (Y, ф) be two IFMTs. Then the function f: X →Y is Continuous if and only if for each IFMT A in X, int[f(A)] ⊆ f[int(A)].

Proof:

Assume that f is continuous.

For any IFMT A in X, int[f(A)] ⊆ f(A) by proposition 4.14.1

⇒ $f^{-1}[int(f(A))] \subseteq A$ by proposition 2.2.b

Since f is continuous and int[f(A)] is open, $f^{-1}[int(f(A))]$ is open. But int(A) is the largest OIFMS

contained in A.

⇒ $f^{-1}[int(f(A))] \subseteq int(A)$

⇒ int[f(A)] ⊆ f[int(A)].

Conversely assume the given condition.

To prove f is continuous, let O be an OIFMS in ф.

Then by assumption, $int[f(f^{-1}(O))] \subseteq f[int(f^{-1}(O))]$

⇒ $int(O) \subseteq f[int(f^{-1}(O))]$

⇒ $O \subseteq f[int(f^{-1}(O))]$ by proposition 4.16

⇒ $f^{-1}(O) \subseteq int(f^{-1}(O))$

ISSN 2229-5518

But by 4.14.1, $int(f^{-1}(O)) \subseteq f^{-1}(O)$

$\Rightarrow f^{-1}(O) = int(f^{-1}(O))$

$\Rightarrow f^{-1}(O)$ is an OIFMS, By proposition 4.11.

$\Rightarrow$ f is continuous.

Hence the proof.

4.22 Theorem Let (X, r) and (Y, ф) be two IFMTs. Then the function f: X →Y is Continuous if and only if for each IFMT B in Y, $f^{-1}[int(B)] \subseteq int[f^{-1}(B)]$

Proof:

Replace A by $f^{-1}(B)$ in theorem 4.21.

Subspace Topology

4.23 Definition Let (X, r) and (Y, ф) be two IFMTs. The topological space Y is called a **subspace** of the topological space X if Y⊆ X and if the open subsets of Y are precisely the subsets O′ of the form

$$O' = O \cap Y$$

for some open subsets O of X. Here we may say that each open subset O′ of Y is the *restriction* to Y of an open subset O of X. O′ is also called **relative open** in Y.

4.24 Example Let X = {1, 2} and define the IFMSs in X as follows.

For n ϵ N^+, p ϵ N

G_n = {<1: (n/n+1, n+1/n+2,…, n+p-1/n+p), (1/n+2, 1/n+3,…,1/n+p+1)>,

<2: (n+1/n+2, n+2/n+3,…,n+p+1/n+p+1), (1/n+3, 1/n+4,…,1/n+p+2)>}

Let r′ = {¬0, ¬1}U{G_n}

Then (X, r′) is a subspace of (X, r) example 4.4.

4.25 Theorem Let (X, r) be an IFMT and let Y be a subset of X. Define the collection ф of subsets of Y as the collection of subsets O′ of Y of the form

$$O' = O \cap Y$$

where O is an OIFMS in (X, r). Then (Y U ¬1, ф) is an IFMT and a subspace of (X, r)

Proof:

We have ¬0 = ¬0 ∩ Y.

Suppose $O'_1, O'_2,\ldots, O'_n$ ϵ ф

Then $O'_i = O_i \cap Y$ for some O_i ϵ r

Then $O'_1 \cap O'_2 \cap \ldots \cap O'_n = (O_1 \cap O_2 \cap \ldots \cap O_n) \cap Y$ ϵ ф

Since $(O_1 \cap O_2 \cap \ldots \cap O_n)$ ϵ r

Finally, suppose that for each α ϵ I, $O'\alpha$ ϵ ф.

Thus for each α ϵ I, $O'\alpha = O\alpha \cap Y$ for some $O\alpha$ ϵ r.

Then $UO'\alpha = U(O\alpha \cap Y) = UO\alpha \cap Y$ ϵ ф since $UO\alpha$ ϵ r

Hence (Y U ¬1, ф) is an IFMT and therefore a subspace of (X, r).

4.26 Theorem Let (Y, ф) be a subspace of the IFMT (X, r). A subset C′ of Y is relatively closed in Y if and only if

$$C' = C \cap Y$$

for some CIFMS C of X.

Proof:

First assume that C′ is relatively closed. Then $\neg_Y(C')$ is relatively open, where $\neg_Y$ denotes the complement w.r.t Y.

Thus $\neg_Y(C') = O \cap Y$ for some OIFMS O in X.

$$\Rightarrow C' = \neg_Y(O \cap Y) = \neg_X(O) \cap Y$$

where $\neg_X(O)$ is a CIFMS in X.

Conversely suppose that $C' = C \cap Y$ for some CIFMS C of X.

Then $\neg_Y(C') = \neg_X(C) \cap Y$.

Hence $\neg_Y(C')$ is relatively open in Y and therefore C′ is relatively closed in Y.

4.26 Definition [8] Let A be a subset of the universal set X. The function i : A →X, which is defined by the correspondence i(x) = x for each x ϵ A is called an *inclusion* mapping or function.

4.27 Theorem Let (Y, ϕ) be a subspace of the IFMT (X, r). Then the inclusion mapping i : Y →X is continuous.

Proof:

For each IFMS A of X, $i^{-1}(A) = A \cap Y$.

Thus if O is an OIFMS in X, then $i^{-1}(O) = O \cap Y$ is a relatively open subset of Y.

Hence i is continuous.

V. CONCLUSION

In this work we studied the topological structures of Intuitionistic Fuzzy Multisets. We introduced the concept of Intuitionistic Fuzzy Multiset Topology. We defined the concept of functions between Intuitionistic Fuzzy Multisets. Open and closed sets are defined. The Closure, Interior, Subspace topology and Continuous functions are defined and their various properties are discussed. The foundations which we made through this paper can be extended and studied to get an insight into the topological structures Intuitionistic Fuzzy Multisets.

REFERENCES

[1] L.A. Zadeh, “Fuzzy sets”, *Inform. and Control* 8 (1965) 338-353.

[2] R.R Yager, “On the theory of bags”, *Int.J. of General Systems* 13 (1986) 23-37.

[3] K.T.Atanassov, “Intuitionistic Fuzzy sets”, *Fuzzy sets and Systems* 20 (1986) 87-96.

[4] Shinoj.T.K, Sunil Jacob John, “Intutionistic fuzzy multisets”, *International Journal of Engineering Science and Innovative Technology*, Volume 2, Issue 6, November 2013.

[5] Shinoj T K, Anagha Baby, Sunil Jacob John, “On Some Algebraic Structures of Fuzzy Multisets”, *Annals of Fuzzy Mathematics and Informatics*, Volume 9, No 1, January 2015,
77-90.

[6] Dogan Coker, “An Introduction to intuitionistic fuzzy topological spaces”, *Fuzzy sets and systems*, 88 (1997), 81-89.

[7] G F Simmons, “Introduction to Topology and Modern Analysis”, McGraw-Hill International Student Edition.

[8] Bert Mendelson, “Introduction to Topology”, 3rd Edition

[9] C. Chang, “Fuzzy Topological spaces”, *J. Math. Anal. Appl.* 24 (1968), 142-145.

[10] K.T.Atanassov, “Intuitionistic Fuzzy sets”, VII ITKRs Session, Sofia (deposed in Central Science-Technical Library of Bulgarian Academy of Science, 1697/84) (in Bulgarian) (1983)

[11] W.D.Blizard, “Dedekind multisets and function shells”, *Theoretical Computer Science*, 110, (1993) 79-98.

[12] W.D.Blizard, “Multiset Theory”, *Notre Dame Journal of Formal Logic*, Vol.30, No.1, (1989) 36-66.

[13] Shinoj.T.K, Sunil Jacob John, “Intuitionistic Fuzzy Multiset and its application in Medical Diagnosis”, *International Journal of Mathematical and Computational Sciences*, World Academy of Science, Engineering and Technology(WASET), Volume 6 (2012) 34-38

[14] Shinoj.T.K, Sunil Jacob John, “Accuracy in Collaborative Robotics: An Intuitionistic FuzzyMultiset Approach”, *Global Journal of Science Frontier Research* (GJSFR), Volume 13 (2013) Issue 10.